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ORTHOGONALITY PROPERTIES OF THE MODES OF THE HELICAL WAVE GUIDE

by

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Title page
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CONTENTS

Article	Page
Abstract	1
1. Introduction	2
2. Mathematical Statement of the Problem	3
3. Orthogonality Relationships	7
4. Proof that γ^2 is always real	17
5. Conclusion	18
References	19

ABSTRACT

This paper contains a study of the orthogonality properties of the modes inside an idealized source free helical wave guide. A vector bi-orthogonal relationship involving the transverse components of E and H is obtained. From this orthogonality property is derived the proof of (i) the roots, γ, of the eigen-value equation are such that γ^2 is always real and (ii) the longitudinal component of the power down the helical guide is the sum of the powers due to each mode alone.

1. Introduction

This paper investigates the orthogonality properties of the modes inside an idealized source free helical wave guide. We assume the guide has cylindrical walls which are perfectly conducting in a helical direction and perfectly non-conducting in the direction perpendicular to this. Such idealizations of the traveling wave tube have been considered by Phillips[1], Brillouin[2] and Harrison[3].

Designate the axis of the cylinder as the z-axis and assume a harmonic time dependence of the form $e^{-i\omega t}$. The field components are found to be a non-separable combination of T.E. and T.M. modes

$$(1.1) \qquad \sum_{n=-\infty}^{\infty} \left[A_n \frac{J_n(\gamma r)}{r} + B_n J_n'(\gamma r) \right] F_n ,$$

(see Stratton[4]) where

$$F_n = e^{in\theta - i\omega t + i\beta z}, \quad \gamma^2 = k^2 - \beta^2 .$$

When the boundary conditions mentioned in the first paragraph are applied at the surface of the guide, there results the two following eigen-value equations for the determination of γ and hence β:

$$(1.2) \qquad \frac{\tan \alpha}{ak} = \frac{n\beta}{k(\gamma a)^2} \pm \frac{J_n'(\gamma a)}{\gamma a J_n(\gamma a)}, \; \beta = \sqrt{k^2 - \gamma^2}, \; n = 0, \pm 1, \pm 2, \cdots .$$

Phillips has shown that Eq. (1.2) has an infinity of real roots γ and at most two pure imaginary roots. One of our principal results is the proof that all values of γ^2 must be real so that Phillips has obtained all possible values of the propagation constant.

The main problem considered in this paper is that of representing an arbitrary transverse field in terms of the modes that propagate down the helix. This is important because it enables us to investigate the response of the traveling wave tube to any type of initial excitation. The customary procedure is to show that the modes form a complete orthonormal set and then use the orthogonality properties of the modes to obtain the representation. However, the modes for the traveling wave tube, either those found by Phillips or those found by Pierce[5], do not possess the usual orthogonality properties. The main object of this paper will be to investigate whether

a more general type of orthogonality can be found for the modes so that a representation theory will still be possible.

It should also be mentioned that investigations have encountered, in a variety of wave guide structures, modes which show similar behaviour. Hansen[7], for the delay line with a reactive wall; Pincherle[8], for a wave guide partially filled with dielectric; and Adler[9] for a circular wave guide with reactive wall have come across the common difficulty of discovering modes which do not possess conventional orthogonality. Furthermore, Phillips and Malin[10] have, in an investigation of the helical wave guide, under modified boundary conditions, discovered that there are only a finite number of discrete modes which clearly means that they do not form a complete set.

For our problem, Eq. (1.2) indicates two sets of modes, each of which is a linear combination of T.E. and T.M. modes, both of which must be present to satisfy the boundary conditions. Because of the peculiar boundary conditions imposed on the guide, the usual orthogonality relationships do not hold. We have been successful in obtaining for the modes of each set a vector bi-orthogonality relationship involving the transverse components of E and H. This relationship is the mathematical counterpart of the following physical fact: the z-component of the power down the guide is the sum of the z-components of the powers due to each mode alone.

2. Mathematical Statement of the Problem

For a monochromatic source the electromagnetic field inside an infinitely long circular cylinder can be represented in the form (Stratton[4])

$$E_r = \sum_{n=-\infty}^{\infty} \left[\frac{i\beta}{\gamma} J_n'(\gamma r) A_n - \frac{\mu\omega n}{\gamma^2 r} J_n(\gamma r) B_n \right] F_n ,$$

(2.1)
$$E_\theta = - \sum_{n=-\infty}^{\infty} \left[\frac{n\beta}{\gamma^2 r} J_n(\gamma r) A_n + \frac{i\mu\omega}{\gamma} J_n'(\gamma r) B_n \right] F_n ,$$

$$E_z = \sum_{n=-\infty}^{\infty} \left[J_n(\gamma r) A_n \right] F_n .$$

(2.2)
$$H_r = \sum_{n=-\infty}^{\infty} \left[\frac{n k^2}{\mu\omega\gamma^2 r} J_n(\gamma r) A_n + \frac{i\beta}{\gamma} J_n'(\gamma r) B_n \right] F_n ,$$

$$H_\theta = \sum_{n=-\infty}^{\infty} \left[\frac{i k^2}{\mu\omega\gamma} J_n'(\gamma r) A_n - \frac{in\beta}{\gamma^2 r} J_n(\gamma r) B_n \right] F_n ,$$

$$H_z = \sum_{n=-\infty}^{\infty} \left[J_n(\gamma r) B_n \right] F_n .$$

In these relations

(2.3) $$\gamma^2 = k^2 - \beta^2 , \quad F_n = \exp(in\theta + i\beta z - i\omega t) , \quad \mathrm{Re}\,\beta > 0 ,$$

and the prime above a Bessel function denotes differentiation with respect to the argument γr.

We shall idealize the helix by assuming that on the surface of the cylinder both the electric field and the magnetic field will vanish in the helical direction. If α denotes the constant angle between the axis of the cylinder and the helical direction, the boundary conditions state that

(2.4)
$$\cos\alpha\, E_\theta + \sin\alpha\, E_z = 0$$
$$\cos\alpha\, H_\theta + \sin\alpha\, H_z = 0$$

at $r = a$ and for all values of θ and z.

Eqs. (2.4) are certainly satisfied if, for each n,

(2.5)
$$\left(\sin\alpha - \frac{n\beta}{\gamma^2 a}\cos\alpha\right) J_n(\gamma a) A_n - \frac{i\mu\omega}{\gamma}\cos\alpha\, J_n'(\gamma a) B_n = 0$$
$$\frac{ik^2\cos\alpha}{\mu\omega\gamma} J_n'(\gamma a) A_n + \left(\sin\alpha - \frac{n\beta\cos\alpha}{\gamma^2 a}\right) J_n(\gamma a) B_n = 0$$

In order that there exist a non-trivial solution for A_n, B_n the determinant of the coefficients of Eqs. (2.5) must vanish; that is

(2.6) $$\left(\sin\alpha - \frac{n\beta}{\gamma^2 a}\cos\alpha\right)^2 J_n^2(a\gamma) = \frac{k^2\cos^2\alpha}{\gamma^2} J_n'^2(\gamma a) ,$$

or

$$\frac{\tan\alpha}{ak} = \frac{n\beta}{k(\gamma a)^2} \pm \frac{J_n'(a\gamma)}{a\gamma J_n(a\gamma)} .$$

If we use Eq. (2.6) together with Eq. (2.5) we find that

(2.7)
$$i \mu \omega B_n = \pm k A_n$$

$$B_n = \mp i A_n \sqrt{\epsilon/\mu} ,$$

the upper sign here corresponding to the upper sign of Eq. (2.6). For n fixed, there will be an infinite number of roots of (2.6). Denote these roots by γ_{mn} ($m = 1, 2, \cdots$). For each value of γ_{mn}, there will exist a mode which we will denote by $E_r^{(mn)}$, $E_\theta^{(mn)}$, etc. These field components may be found by using Eq. (2.7) in (2.1) and (2.2). We have, after omitting a constant factor,

(2.8)
$$E_r^{(mn)} = \left[\mp \frac{\beta_{mn}}{\gamma_{mn}} J_n' (\gamma_{mn} r) - \frac{nk}{\gamma^2_{mn} r} J_n (\gamma_{mn} r)\right] e^{in\theta} \quad e^{i \beta_{mn} z} ,$$

$$E_\theta^{(mn)} = \left[\mp \frac{in\beta_{mn}}{\gamma^2_{mn} r} J_n (\gamma_{mn} r) - \frac{ik}{\gamma_{mn}} J_n' (\gamma_{mn} r)\right] e^{in\theta} \quad e^{i \beta_{mn} z}$$

$$E_z^{(mn)} = \pm i J_n (\gamma_{mn} r) e^{in\theta} \quad e^{i \beta_{mn} z}$$

$$H_r^{(mn)} = \left[\pm \frac{in}{\gamma^2_{mn} r} J_n (\gamma_{mn} r) + \frac{\beta_{mn}}{\gamma_{mn}} J_n' (\gamma_{mn} r)\right] e^{in\theta} \quad e^{i \beta_{mn} z}$$

$$H_\theta^{(mn)} = \left[\mp \frac{k}{\gamma_{mn}} J_n' (\gamma_{mn} r) - \frac{n\beta_{mn}}{\gamma^2_{mn} r} J_n (\gamma_{mn} r)\right] e^{in\theta} \quad e^{i \beta_{mn} z}$$

$$H_z^{(mn)} = \left[J_n(\gamma_{mn} r)\right] e^{in\theta} \quad e^{i \beta_{mn} z}$$

These field equations correspond to linear combinations of the familiar T.E. and T.M. modes present in ordinary wave guides.

Our main interest is in determining the field produced in the tube by some excitation at an arbitrary cross section, say $z = 0$. Suppose then, we are given E_r and E_θ at $z = 0$, how can the values of E_r and E_θ in the rest of the tube be determined?

If the set of modes as given by Eq. (2.8) is complete then any arbitrary field will be a linear combination of the fields in Eq. (2.8), that is, there will exist constants C_{mn} such that

$$
\begin{aligned}
E_r &= \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} C_{mn} E_r^{(mn)} \\
E_\theta &= \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} C_{mn} E_\theta^{(mn)}
\end{aligned}
\tag{2.9}
$$

How can the constants C_{mn} be found? Let us represent the values of E_r and E_θ at $z=0$ by $E_r^{(0)}$, $E_\theta^{(0)}$. Then putting $z=0$ in Eqs. (2.9) and (2.8), we find that

$$
\begin{aligned}
E_r^{(0)} &= \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} C_{mn} \left[\frac{\beta_{mn}}{\gamma_{mn}} J_n'(\gamma_{mn} r) \pm \frac{nk}{\gamma_{mn}^2 r} J_n(\gamma_{mn} r) \right] e^{in\theta} \\
E_\theta^{(0)} &= \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} C_{mn} \left[\frac{in\beta_{mn}}{\gamma_{mn}^2 r} J_n(\gamma_{mn} r) \pm \frac{ik}{\gamma_{mn}} J_n'(\gamma_{mn} r) \right] e^{in\theta}
\end{aligned}
\tag{2.10}
$$

Using the well known orthogonality property of the trigonometric functions, if we multiply Eq. (2.10) by $e^{-in\theta}\, d\theta$ and integrate from 0 to 2π, we find that

$$
\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} E_r^{(0)} e^{-in\theta}\, d\theta &= \sum_{m=1}^{\infty} C_{mn} \left[\frac{\beta_{mn}}{\gamma_{mn}} J_n'(\gamma_{mn} r) \pm \frac{nk}{\gamma_{mn}^2 r} J_n(\gamma_{mn} r) \right] \\
\frac{1}{2\pi} \int_0^{2\pi} E_\theta^{(0)} e^{-in\theta}\, d\theta &= \sum_{m=1}^{\infty} C_{mn} \left[\frac{in\beta_{mn}}{\gamma_{mn}^2 r} J_n(\gamma_{mn} r) \pm \frac{ik}{\gamma_{mn}} J_n'(\gamma_{mn} r) \right]
\end{aligned}
\tag{2.11}
$$

In Eq. (2.11), the left hand side is a known function of r and the constants C_{mn} must be determined.

Now, if the functions in the brackets formed an orthogonal set, we could perform an integration, just as we did in going from (2.10) to (2.11), and express C_{mn} as a double integral involving $E_r^{(0)}$ and $E_\theta^{(0)}$. In the next section we shall investigate the orthogonality properties of these functions.

3. Orthogonality Relationships

For the examination of the orthogonality properties we split off from E and H the harmonic time dependence and the exponential z-dependence and designate the new quantities by means of the old symbols E and H. This does not cause any confusion because the integrals which we will employ are with respect to the transverse coordinates only. Our only purpose in doing this is to avoid writing down the (constant) factor $e^{i\beta z - i\omega t}$.

We now resolve the vectors E and H into longitudinal and transverse components

$$E = i_z E_z + E_T \tag{3.1}$$
$$H = i_z E_z + H_T$$

In Eq. (3.1), i_z is the unit vector in the longitudinal direction, E_z is a scalar and E_T a vector, function depending only on the transverse coordinates. Let i_r and i_θ be unit vectors in the radial and tangential directions, put

$$e_r^{(mn)}(r,\theta,\gamma,n) = e_r^{(mn)} \equiv \left[\mp \frac{\beta_{mn}}{\gamma_{mn}} J_n'(\gamma_{mn} r) - \frac{kn}{\gamma_{mn}^2 r} J_n(\gamma_{mn} r)\right]\eta e^{in\theta},$$

$$e_\theta^{(mn)}(r,\theta,\gamma,n) = e_\theta^{(mn)} \equiv \left[\mp \frac{in\beta_{mn}}{\gamma_{mn}^2 r} J_n(\gamma_{mn} r) - \frac{ik}{\gamma_{mn}} J_n'(\gamma_{mn} r)\right]\eta e^{in\theta},$$

(3.2)

$$h_r^{(mn)}(r,\theta,\gamma,n) = h_r^{(mn)} \equiv \left[\pm \frac{ink}{\gamma_{mn}^2 r} J_n(\gamma_{mn} r) + \frac{i\beta_{mn}}{\gamma_{mn}} J_n'(\gamma_{mn} r)\right] e^{in\theta},$$

$$h_\theta^{(mn)}(r,\theta,\gamma,n) = h_\theta^{(mn)} \equiv \left[\mp \frac{k}{\gamma_{mn}} J_n'(\gamma_{mn} r) - \frac{n\beta_{mn}}{\gamma_{mn}^2 r} J_n(\gamma_{mn} r)\right] e^{in\theta},$$

$$e_z^{(mn)}(r,\theta,\gamma,n) = e_z^{(mn)} \equiv \left[\mp J_n(\gamma_{mn} r)\right]\eta\, e^{in\theta},$$

$$h_z^{(mn)}(r,\theta,\gamma,n) = h_z^{(mn)} \equiv \left[J_n(\gamma_{mn} r)\right] e^{in\theta};$$

then

$$E_T = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} (i_r e_r^{(mn)} + i_\theta e_\theta^{(mn)}) C_{mn} = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} e_T^{(mn)} C_{mn} ,$$

(3.3)

$$H_T = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} (i_r h_r^{(mn)} + i_\theta h_\theta^{(mn)}) C_{mn} = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} h_T^{(mn)} C_{mn} ,$$

$$E_z = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} e_z^{(mn)} C_{mn} .$$

We shall now investigate different types of orthogonality conditions involving $e_T^{(mn)}$, $h_T^{(mn)}$, $e_z^{(mn)}$, $h_z^{(mn)}$. In conventional wave guides the longitudinal components of the modes are orthogonal, that is

(3.4) $$\int_A e_z^{(mn)} e_z^{(m_1 n_1)} d\sigma = 0$$

if $(m-m_1)^2 + (n-n_1)^2 \neq 0$, where the integral of Eq. (3.4) is taken over the cross-sectional area A of the cylinder. If the cross section is a circle, Eq. (3.4) becomes

$$\int_0^{2\pi} d\theta \int_0^a r\, e_z^{(mn)} e_z^{(m_1 n_1)} dr = 0$$

and for the case of the traveling wave tube, where $e_z^{(mn)}$ is given by Eq. (3.2) the orthogonality condition reduces to

(3.5) $$\int_0^{2\pi} d\theta \int_0^a r\, J_n(\gamma_{mn} r)\, J_{n_1}(\gamma_{m_1 n_1} r)\, e^{in\theta}\, e^{in_1\theta}\, dr$$

If $n \neq -n_1$, the integration with respect to θ gives zero. If $n = -n_1$, the value of the integral in Eq. (3.5) depends on

$$\int_0^a r\, J_n(\gamma_{mn} r)\, J_{-n}(\gamma_{m_1, -n} r)\, dr$$

$$= (-1)^n \int_0^a r\, J_n(\gamma_{mn} r)\, J_n(\gamma_{m_1, n} r)\, dr$$

since

$$J_n(x) = (-1)^n J_{-n}(x) .$$

The integral may be evaluated by formula 8 page 134 in Watson's <u>Bessel Functions</u>. We find that it equals

$$\frac{(-1)^n a}{\gamma^2_{mn} - \gamma^2_{m_1n}} \left[\gamma_{m_1n} \frac{J_n'(\gamma_{m_1n}a)}{J_n(\gamma_{m_1n}a)} - \gamma_{mn} \frac{J_n'(\gamma_{mn}a)}{J_n(\gamma_{mn}a)} \right] J_n(\gamma_{mn}a)\, J_n(\gamma_{m_1n}a)$$

$$= \frac{1}{\gamma^2_{mn} - \gamma^2_{m_1n}} \left[\frac{\tan\alpha}{k} (\gamma_{m_1n} - \gamma_{mn}) + \frac{n}{ak} \left[\frac{\beta_{m_1n}}{\gamma_{m_1n}} + \frac{\beta_{mn}}{\gamma_{mn}} \right] J_n(\gamma_{mn}a)\, J_n(\gamma_{m_1n}a) \right.$$

by use of Eq. (2.6). In general, there is no reason for this to be zero and by actual calculation it can be shown to be different from zero for particular values of m, n, m_1. A similar evaluation shows that for the traveling wave tube none of the integrals

$$\int_A h_z^{(mn)}\, h_z^{(m_1n)}\, d\sigma, \quad \int_A e_z^{(mn)}\, \bar{e}_z^{(m_1n)}\, d\sigma \text{ or } \int_A h_z^{(mn)}\, \bar{h}_z^{(m_1n)}\, d\sigma$$

is zero. Here $\bar{e}_z$ and $\bar{h}_z$ mean the conjugates of e_z and h_z.

We next investigate the possibility of an orthogonality condition involving the transverse components of the field. An evaluation of the integrals involved shows that

$$(\gamma^2_{mn} - \gamma^2_{m_1n}) \int_A e_T^{(mn)} \cdot \bar{e}_T^{(m_1n)}\, d\sigma$$

$$= \frac{n}{k} (\beta_{mn} - \beta_{m_1n})\, J_n(\gamma_{mn}a)\, J_n(\bar{\gamma}_{m_1n}a)$$

which, again, is in general not equal to zero.

It will be proved that

$$\int_A i_z \cdot e_T^{(mn)} \times \bar{h}_T^{(m_1n_1)}\, d\sigma = 0 \tag{3.6}$$

for $(m-m_1)^2 + (n-n_1)^2 \neq 0$. Just as before, if $n \neq n_1$, the θ integration will make the integral in Eq. (3.6) equal to zero. Suppose now that $n = n_1$, then from Eqs. (3.3) and (3.2)

$$i_z \cdot \left(e_T^{(mn)} \times \bar{h}_T^{(m_1 n)}\right) = e_r^{(mn)} \bar{h}_\theta^{(m_1 n)} - e_\theta^{(mn)} \bar{h}_r^{(m_1 n)}$$

$$= \frac{\eta}{\gamma_{mn}\gamma_{m_1 n}}\Big[(\beta_{mn}+\beta_{m_1 n})\Big\{k\, J_n'(\gamma_{mn} r) J_n'(\gamma_{m_1 n} r) + \frac{k\, n^2}{\gamma_{mn}\gamma_{m_1 n} r^2} J_n(\gamma_{mn} r) J_n(\gamma_{m_1 n} r)\Big\}$$

(3.7)

$$+ \frac{n(k^2+\beta_{mn}\beta_{m_1 n})}{r\gamma_{mn}\gamma_{m_1 n}}\Big\{\gamma_{mn} J_n'(\gamma_{mn} r) J_n(\gamma_{m_1 n} r) + \gamma_{m_1 n} J_n(\gamma_{mn} r) J_n'(\gamma_{m_1 n} r)\Big\}\Big]$$

Hence, by Eq. (3.7)

$$\int_A i_z \cdot \left(e_T^{(mn)} \times \bar{h}_T^{(m_1 n)}\right) d\sigma = \int_0^{2\pi} d\theta \int_0^a r\, i_z \cdot \left(e_T^{(mn)} \times \bar{h}_T^{(m_1 n)}\right) dr$$

$$= \frac{2\pi\eta}{\gamma_{mn}\gamma_{m_1 n}} (\beta_{mn} + \beta_{m_1 n})\, k \int_0^a r\, J_n'(\gamma_{mn} r)\, J_n'(\gamma_{m_1 n} r)\, dr$$

(3.8)

$$+ \frac{2\pi\eta}{(\gamma_{mn}\gamma_{m_1 n})^2} kn^2 (\beta_{mn}+\beta_{m_1 n}) \int_0^a \frac{J_n(\gamma_{mn} r)\, J_n(\gamma_{m_1 n} r)}{r} dr$$

$$+ 2\pi\eta n \frac{(k^2 + \beta_{mn}\beta_{m_1 n})}{\gamma_{mn}\gamma_{m_1 n}} \int_0^a \Big\{\gamma_{mn} J_n'(\gamma_{mn} r)\, J_n(\gamma_{m_1 n} r)$$

$$+ \gamma_{m_1 n} J_n'(\gamma_{m_1 n} r)\, J_n(\gamma_{mn} r)\Big\} dr$$

Designating the three integrals which appear in Eq. (3.8) by I_1, I_2 and I_3, that equation may be rewritten as

$$\int_A i_z \cdot \left(e_T^{(mn)} \times \bar{h}_T^{(m_1 n)}\right) d\sigma$$

(3.9)

$$= \frac{2\pi\eta}{\gamma_{mn}\gamma_{m_1 n}} k\left(\beta_{mn}+\beta_{m_1 n}\right) I_1 + \frac{2\pi\eta}{(\gamma_{mn}\gamma_{m_1 n})^2} kn^2\left(\beta_{mn}+\beta_{m_1 n}\right) I_2$$

$$+ \frac{2\pi\eta n}{(\gamma_{mn}\gamma_{m_1 n})^2}\left(k^2+\beta_{mn}\beta_{m_1 n}\right) I_3$$

However

(3.10) $$I_3 = J_n(\gamma_{mn}a)\, J_n(\gamma_{m_1 n}a)$$

Moreover an integration by parts shows that

$$I_1 = a\,\frac{J_n(\gamma_{mn}a)\, J_n'(\gamma_{m_1 n}a)}{\gamma_{mn}} + \frac{\gamma_{m_1 n}}{\gamma_{mn}} \int_0^a r\, J_n(\gamma_{mn}r)\, J_n(\gamma_{m_1 n}r)\, dr$$

(3.11)

$$- \frac{n^2}{\gamma_{mn}\gamma_{m_1 n}} I_2 \quad .$$

or see Watson, Bessel Functions, p. 134.

$$I_1 = a\frac{J_n(\gamma_{mn}a)J_n'(\gamma_{m_1 n}a)}{\gamma_{mn}} + \frac{\gamma_{m_1 n}}{\gamma_{mn}}\,\frac{a}{\gamma_{mn}^2-\gamma_{m_1 n}^2}\left[\gamma_{m_1 n}J_n'(\gamma_{m_1 n}a)J_n(\gamma_{mn}a)-\gamma_{mn}J_n'(\gamma_{mn}a)J_n(\gamma_{m_1 n}a)\right]$$

(3.12)

$$- \frac{n^2}{\gamma_{mn}\gamma_{m_1 n}} I_2 \quad .$$

so that if the terms in Eq. (3.12) free of I_2 be combined, we have

$$I_1 = \frac{a}{\gamma_{mn}^2 - \gamma_{m_1n}^2}\left[\gamma_{mn} J_n(\gamma_{mn}) J_n'(\gamma_{m_1n}a) - \gamma_{m_1n} J_n(\gamma_{m_1n}a) J_n(\gamma_{mn}a)\right]$$

(3.13)

$$- \frac{n^2}{\gamma_{mn}\gamma_{m_1n}} I_2 \quad .$$

Using Eqs. (3.13) and (3.10) in Eq. (3.9), there results

$$\int_A i_z \cdot \left(e_T^{(mn)} \times \bar{h}_T^{(m_1n)}\right) d\sigma$$

$$= \frac{2\pi\eta}{\gamma_{mn}\gamma_{m_1n}}\left[ka \frac{(\beta_{mn}+\beta_{m_1n})}{\gamma_{mn}^2 - \gamma_{m_1n}^2}\left(\gamma_{mn} J_n(\gamma_{mn}a) J_n'(\gamma_{m_1n}a) - \gamma_{m_1n} J_n(\gamma_{m_1n}a) J_n'(\gamma_{mn}a)\right)\right.$$

(3.14)

$$\left. + n \frac{(k^2+\beta_{mn}\beta_{m_1n})}{\gamma_{mn}\gamma_{m_1n}} J_n(\gamma_{mn}a) J_n(\gamma_{m_1n}a)\right] .$$

$$= \frac{2\pi\eta}{\gamma_{mn}\gamma_{m_1n}}\left[k a^2 \frac{(\beta_{mn}+\beta_{m_1n})}{\gamma_{mn}^2 - \gamma_{m_1n}^2} \gamma_{mn}\gamma_{m_1n} J_n(\gamma_{mn}a) J_n(\gamma_{m_1n}a) \left\{ \frac{J_n'(\gamma_{m_1n}a)}{\gamma_{m_1n}a\, J_n(\gamma_{m_1n}a)}\right.\right.$$

$$\left.\left. - \frac{J_n'(\gamma_{mn}a)}{\gamma_{mn}a\, J_n(\gamma_{mn}a)}\right\} + n \frac{(k^2+\beta_{mn}\beta_{m_1n})}{\gamma_{mn}\gamma_{m_1n}} J_n(\gamma_{mn}a) J_n(\gamma_{m_1n}a)\right]$$

The eigenvalue equation (2.6) reduces the right hand side of Eq. (3.14) to

$$\frac{2\pi\eta}{\gamma_{mn}\gamma_{m_1n}}\Bigg[\; k a^2 \frac{(\beta_{mn}+\beta_{m_1n})}{\gamma^2_{mn}-\gamma^2_{m_1n}}\gamma_{mn}\gamma_{m_1n}J_n(\gamma_{mn}a)J_n(\gamma_{m_1n})\Bigg\{\frac{\tan\alpha}{ak}-\frac{n\beta_{m_1n}}{(\gamma_{m_1n}a)^2 k}$$

$$-\frac{\tan\alpha}{ak}+\frac{n\beta_{mn}}{(\gamma_{mn}a)^2 k}\Bigg\} + n(k^2+\beta_{mn}\beta_{m_1n})\, J_n(\gamma_{mn}a)\, J_n(\gamma_{m_1n}a)\Bigg]$$

$$=\frac{2\pi\eta}{\gamma_{mn}\gamma_{m_1n}}\, n\, J_n(\gamma_{mn}a)\, J_n(\gamma_{m_1n}a)\Bigg[\frac{\beta_{mn}+\beta_{m_1n}}{\gamma^2_{mn}-\gamma^2_{m_1n}}\gamma_{mn}\gamma_{m_1n}\left(\frac{\beta_{mn}}{\gamma^2_{mn}}-\frac{\beta_{m_1n}}{\gamma^2_{m_1n}}\right)$$

$$+\frac{k^2+\beta_{mn}\beta_{m_1n}}{\gamma_{mn}\gamma_{m_1n}}\Bigg]$$

(3.15)

$$=\frac{2\pi\eta}{(\gamma_{mn}\gamma_{m_1n})^2}\;\frac{n}{\gamma^2_{mn}-\gamma^2_{m_1n}}J_n(\gamma_{mn}a)J_n(\gamma_{m_1n}a)(\beta_{mn}+\beta_{m_1n})\Big[(\beta_{m_1n}-\beta_{mn})(k^2+\beta_{mn}\beta_{m_1n})$$

$$+\beta_{mn}\gamma^2_{m_1n}-\beta_{m_1n}\gamma^2_{mn}\Big]$$

$$=\frac{2\pi\eta}{(\gamma_{mn}\gamma_{m_1n})^2}\;\frac{\beta_{mn}+\beta_{m_1n}}{\gamma^2_{mn}-\gamma^2_{m_1n}}\, n\, J_n(\gamma_{mn}a)\, J_n(\gamma_{m_1n}a)\Big[\, k^2\beta_{m_1n}-k^2\beta_{mn}$$

$$+\beta^2_{m_1n}\beta_{mn}-\beta^2_{mn}\beta_{m_1n}+\beta_{mn}k^2-\beta_{mn}\beta^2_{m_1n}$$

$$-\beta_{m_1n}k^2+\beta_{m_1n}\beta^2_{mn}\Big]$$

$$=0$$

Hence

$$\int_A i_z \cdot \left(e_T^{(mn)} \times \bar{h}_T^{(m_1 n)}\right) d\sigma = 0. \tag{3.16}$$

A calculation similar to the above shows that

$$\int_A i_z \cdot \left(e_T^{(mn)} \times \bar{h}_T^{(mn)}\right) d\sigma = \frac{4\pi\eta}{\gamma_{mn}^2}\left\{\frac{ka\,\beta_{mn}}{\gamma_{mn}} J_n(\gamma_{mn}a)\, J_n'(\gamma_{mn}a) + \frac{n(k^2+\beta_{mn}^2)}{2\,\gamma_{mn}^2} J_n^2(\gamma_{mn}a)\right.$$

$$\left. + \frac{1}{2}\, k\,\beta_{mn} a^2 \left[J_n'^2(\gamma_{mn}a) + \left(1 - \frac{n^2}{(\gamma_{mn}a)^2}\right) J_n^2(\gamma_{mn}a)\right]\right\} \tag{3.17}$$

The equation (2.6) really gives rise to two sets of modes, one for the roots derived with the plus sign, one for the roots with minus sign. As has been proved the orthogonality condition (3.16) holds for each set separately. It would be highly desirable for Eq. (3.16) to hold for γ_{mn} a root of Eq. (2.6) with plus sign and $\gamma_{m_1 n}$ a root of Eq.(2.6) with minus. An actual evolution shows this to be false. Possibly some other type of orthogonality condition would show this. This question should be studied since the usual wave guide structures show no analogous behaviour.

The longitudinal component of power down the tube resulting from two modes γ_{mn} and $\gamma_{m_1 n}$ is given by

$$\mathrm{Re} \int_A i_z \cdot \left(e_T^{(mn)} + e_T^{(m_1 n)}\right) \times \left(\bar{h}_T^{(mn)} + \bar{h}_T^{(m_1 n)}\right) d\sigma$$

$$= \mathrm{Re} \int_A \left(i_z \cdot e_T^{(mn)} \times \bar{h}_T^{(mn)} + i_z \cdot e_T^{(mn)} \times \bar{h}_T^{(m_1 n)}\right. \tag{3.18}$$

$$\left. + i_z \cdot e_T^{(m_1 n)} \times \bar{h}_T^{(mn)} + i_z \cdot e_T^{(m_1 n)} \times \bar{h}_T^{(m_1 n)}\right) d\sigma \quad .$$

If γ_{mn}, γ_{m_1n} are two roots of Eq. (2.6) with the same sign, the result of Eq. (3.16) shows that the middle integrals of Eq. (3.18) are individually zero. If now γ_{mn} and γ_{m_1n} are two roots of Eq. (2.6) one derived with plus sign, the other with minus sign, we have with the aid of Eq. (3.2)

$$\iint_A \left\{ i_z \cdot e_T^{(mn)} \times \bar{h}_T^{(m_1n)} + i_z \cdot e_T^{(m_1n)} \times \bar{h}_T^{(mn)} \right\} d\sigma$$

$$= \eta \iint_A i_z \cdot \left[\left(-\frac{\beta_{mn}}{\gamma_{mn}} J_n'(\gamma_{mn} r) - \frac{kn}{\gamma_{mn}^2 r} J_n(\gamma_{mn} r) \right) i_r + \left(\frac{-in\beta_{mn}}{\gamma_{mn}^2 r} J_n(\gamma_{mn} r) - \frac{ik}{\gamma_{mn}} J_n'(\gamma_{mn} r) \right) i_\theta \right] \times$$

(3.19)

$$\left[\left(\frac{ink}{\gamma_{m_1n}^2 r} J_n(\gamma_{m_1n} r) - \frac{i\beta_{m_1n}}{\gamma_{m_1n}} J_{m_1n}'(\gamma_{m_1n} r) \right) i_r + \left(\frac{k}{\gamma_{m_1n}} J_n'(\gamma_{m_1n} r) - \frac{n\beta_{m_1n}}{\gamma_{m_1n}^2 r} J_n(\gamma_{m_1n} r) \right) i_\theta \right]$$

$$+ i_z \cdot \left[\left(\frac{\beta_{m_1n}}{\gamma_{m_1n}} J_n'(\gamma_{m_1n} r) - \frac{kn}{\gamma_{m_1n}^2 r} J_n(\gamma_{m_1n} r) \right) i_r + \left(\frac{in\beta_{m_1n}}{\gamma_{m_1n}^2 r} J_n(\gamma_{m_1n} r) - \frac{ik}{\gamma_{m_1n}} J_n'(\gamma_{m_1n} r) \right) i_\theta \right] \times$$

$$\left[\left(\frac{-ink}{r\gamma_{mn}^2} J_n(\gamma_{mn} r) - \frac{i\beta_{mn}}{\gamma_{mn}} J_n'(\gamma_{mn} r) \right) i_r + \left(-\frac{k}{\gamma_{mn}} J_n'(\gamma_{mn} r) - \frac{n\beta_{mn}}{r\gamma_{mn}^2} J_n(\gamma_{mn} r) \right) i_\theta \right] \Bigg\} d\sigma =$$

$$
\begin{aligned}
= \eta \int_A \Big[& - \frac{k \beta_{mn}}{\gamma_{mn} \gamma_{m_1 n}} J'_n(\gamma_{mn} r) J'_n(\gamma_{m_1 n} r) + \frac{n \beta_{mn} \beta_{m_1 n}}{\gamma_{mn} \gamma^2_{m_1 n} r} J'_n(\gamma_{mn} r) J_n(\gamma_{m_1 n} r) \\
& - \frac{k^2 n}{\gamma^2_{mn} \gamma_{m_1 n} r} J_n(\gamma_{mn} r) J'_n(\gamma_{m_1 n} r) + \frac{k n^2 \beta_{m_1 n}}{(\gamma_{mn} \gamma_{m_1 n})^2 r^2} J_n(\gamma_{mn} r) J_n(\gamma_{m_1 n} r) \\
& - \frac{k n^2 \beta_{mn}}{(\gamma_{mn} \gamma_{m_1 n})^2 r^2} J_n(\gamma_{mn}) J_n(\gamma_{m_1 n} r) + \frac{n \beta_{mn} \beta_{m_1 n}}{\gamma_{m_1 n} \gamma^2_{mn} r} J_n(\gamma_{mn} r) J'_n(\gamma_{m_1 n} r) \\
& - \frac{k^2 n}{r \gamma^2_{m_1 n} \gamma_{mn}} J'_n(\gamma_{mn} r) J_n(\gamma_{m_1 n} r) + \frac{k \beta_{m_1 n}}{\gamma_{mn} \gamma_{m_1 n}} J'_n(\gamma_{m_1 n} r) J'_n(\gamma_{mn} r)
\end{aligned}
$$

(3.19)

$$
\begin{aligned}
& - \frac{k \beta_{m_1 n}}{\gamma_{m_1 n} \gamma_{mn}} J'_n(\gamma_{mn} r) J'_n(\gamma_{mn} r) - \frac{n \beta_{m_1 n} \beta_{mn}}{r \gamma_{m_1 n} \gamma^2_{mn}} J_n(\gamma_{m_1 n} r) J'_n(\gamma_{m_1 n} r) \\
& + \frac{k^2 n}{r \gamma^2_{m_1 n} \gamma_{mn}} J_n(\gamma_{m_1 n} r) J'_n(\gamma_{mn} r) + \frac{k n^2 \beta_{mn}}{r^2 (\gamma_{m_1 n} \gamma_{mn})^2} J_n(\gamma_{m_1 n} r) J_n(\gamma_{mn} r) \\
& - \frac{n^2 k \beta_{m_1 n}}{r^2 (\gamma_{mn} \gamma_{m_1 n})^2} J_n(\gamma_{mn} r) J_n(\gamma_{m_1 n} r) - \frac{n \beta_{mn} \beta_{m_1 n}}{r \gamma^2_{m_1 n} \gamma_{mn}} J_n(\gamma_{m_1 n} r) J'_n(\gamma_{mn} r) \\
& - \frac{k^2 n}{r \gamma^2_{mn} \gamma_{m_1 n}} J_n(\gamma_{mn} r) J'_n(\gamma_{m_1 n} r) + \frac{k \beta_{mn}}{\gamma_{mn} \gamma_{m_1 n}} J'_n(\gamma_{mn} r) J'_n(\gamma_{m_1 n} r) \Big] \, d\sigma
\end{aligned}
$$

$$= 0$$

Hence in either case

$$\mathrm{Re}\int_A i_z \cdot (e_T^{(mn)} + e_T^{(m_1 n)}) \times (\bar{h}_T^{(mn)} + \bar{h}_T^{(m_1 n)})\; d\sigma$$

(3.20)

$$= \mathrm{Re}\int\!\!\int_A \left\{ i_z \cdot e_T^{(mn)} \times \bar{h}_T^{(mn)} + i_z \cdot e_T^{(m_1 n)} \times \bar{h}_T^{(m_1 n)} \right\} d\sigma$$

Eq. (3.20) may be translated into the important physical result: the longitudinal component of the power down the helical guide is the sum of the powers due to each mode alone.

4. Proof that γ^2 is always real.

The orthogonality condition of Section 3 can be used to prove that γ^2 is always real. Since $J_n(\gamma r)$ satisfies a second order differential equation with real coefficients it follows that

(4.1) $$J_n(\bar{\gamma} a) = \bar{J}_n(\gamma a),\; J_n'(\bar{\gamma} a) = \bar{J}_n'(\gamma a).$$

It has been shown that

(4.2) $$(\gamma^2_{mn} - \gamma^2_{m_1 n}) \int_A i_z \cdot (e_T^{(mn)} \times \bar{h}_T^{(m_1 n)})\; d\sigma = 0$$

for $\gamma_{mn} \mp \gamma_{m_1 n}$, where γ_{mn}, $\gamma_{m_1 n}$ are two arbitrary roots of Eq. (2.6) with the same sign. In Eq. (4.2) replace $\gamma_{m_1 n}$ by $\bar{\gamma}_{m,n}$; the right hand side is zero; the left hand side — see Eq. (3.8)

$$\frac{2\pi\eta}{|\gamma_{mn}|^2}(\gamma^2_{mn} - \bar{\gamma}^2_{mn})\Big[(\beta_{mn} + \bar{\beta}_{mn})\,k\int_0^a r\,J_n'(\gamma_{mn} r)\,\bar{J}_n'(\gamma_{mn} r)\,dr$$

(4.3)

$$+ \frac{k\,n^2}{|\gamma_{mn}|^2}(\beta_{mn} + \bar{\beta}_{mn})\int_0^a \frac{J_n(\gamma_{mn} r)\bar{J}_n(\gamma_{mn} r)}{r}\,dr$$

$$+ n\,\frac{(k^2 + |\beta_{mn}|^2)}{|\gamma_{mn}|^2}\,J_n(\gamma_{mn} a)\,\bar{J}_n(\gamma_{mn} a)\Big] =$$

$$= \frac{2\pi\eta}{|\gamma_{mn}|^2}(\gamma^2_{mn} - \bar{\gamma}^2_{mn})\Big[(\beta_{mn} + \bar{\beta}_{mn})\, k\int_0^a r\,|J'_n(\gamma_{mn} r)|^2\, dr$$

$$+ \frac{k\, n^2}{|\gamma_{mn}|^2}(\beta_{mn} + \bar{\beta}_{mn})\int_0^a \left|\frac{J_n(\gamma_{mn} r)}{r}\right|^2 dr \qquad (4.3)$$

$$+ n\,\frac{(k^2 + |\beta|^2)}{|\gamma_{mn}|^2}\,|J_n(\gamma_{mn} a)|^2\Big]\ .$$

If we recall that $\mathrm{Re}\,\beta > 0$, it is seen that all the terms in the bracket of Eq. (4.3) are positive, thus $\gamma^2_{mn} - \bar{\gamma}^2_{mn}$ must be zero, from which we infer that

$$\gamma_{mn} = \bar{\gamma}_{mn}$$

Hence the eigen-value equation (2.6) cannot have any complex roots. This result complements the work of Phillips and it may now be stated that Eq. (2.6) has an infinity of real roots, a finite (zero) number of pure imaginary roots and no complex roots.

5. Conclusion

We have been successful in finding an orthogonality condition for each set of modes in the helical wave guide under the boundary conditions of this paper. From the orthogonality condition we derived two conclusions: (i) the z-component of power down the tube is the sum of the powers due to each mode alone; (ii) the roots of the eigen-value equation giving the mode structure cannot be complex.

The orthogonality condition is restricted however to modes given by the same transcendental equation. The condition fails for two modes, each of which is obtained from a different equation. This is in sharp contrast to the conventional wave guide problem where it is not uncommon to derive two eigen-value equations for the modes, and yet orthogonality exists between mixed modes.

The failure of our condition means that there is no way of expanding an arbitrary function in terms of the mode functions based on Eq. (3.6). The question needs further study; in particular, possibly, a new orthogonality condition.

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